NASA TECHNICAL TRANSLATION NASA TT F-14480

RELATION BETWEEN SYSTEMS OF POLYNOMIALS ORTHOGONAL WITH RESPECT TO VARIOUS DISTRIBUTION FUNCTIONS

by
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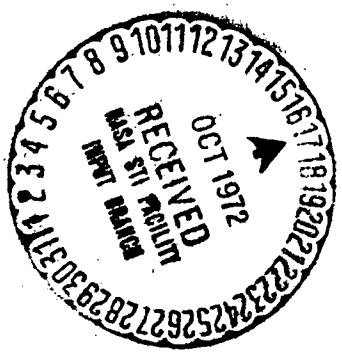
(NASA-TT-F-14480) RELATION BETWEEN SYSTEMS OF POLYNOMIALS ORTHOGONAL WITH RESPECT TO VARIOUS DISTRIBUTION FUNCTIONS V.B. Uvarov (NASA) Mar. 1972 21 p
N72-32592
CSCL 12A G3/19 43745
Unclas

Translation of "O svyazi sistem polinomov, ortogonal'nykh otnositel'no razlichnykh funktsiy raspredeleniya", Vychislitel'naya matematika i matematicheskaya fizika(Computational Mathematics and Mathematical Physics), Vol. 9, No. 6, (Nov-Dec 1969), pp. 1253-1262.



NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, D.C. 20546 MARCH 1972

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Moscow

Translated (June 1970) from the Russian "*O svyazi sistem polinomov, ortogonal'nykh otnositel'no razlichnykh funktsiĭ raspredeleniya,*" Vychislitel'naya matematika i matematicheskaya fizika, vol. 9, no. 6 (Nov-Dec 1969), pp. 1253-1262.

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It is known that the x_k are zeros of polynomial $P_n(x)$, orthogonal with respect to the function $\rho(x)$, i.e. of a polynomial for which

$$\int_{-\infty}^{\infty} P_n(x) x^m \, d\rho(x) = 0 \quad \text{when} \quad m < n.$$

The constants a_k are also easily expressed in terms of polynomials $P_n(x)$. If

$$c_k = \int_{-\infty}^{\infty} x^k \, d\rho(x),$$

then $P_n(x) = b_n \det \|a_{ij}\|$, $1 \leqslant i, j \leqslant n+1$, where $a_{ij} = c_{i+j-2}$ when $i < n+1$ and $a_{n+1,\,j} = x^{j-1}$.

This is not a convenient formula when n is large. The polynomials $P_n(x)$ are easily computable only for a small class of distribution functions $\rho(x)$, when the coefficients of recurrent relation are known for $P_n(x)$ (cf. Section 1). The present paper aims to solve the following problem: how to find polynomials $\overline{P}_n(x)$ orthogonal with respect to the distribution function $\overline{\rho}(x)$, relatively simply expressed in terms of $\rho(x)$, from polynomials $P_n(x)$ orthogonal with respect to the distribution function $\rho(x)$. Even apart from its practical applications, this problem is of considerable theoretical interest.

In Sections 1 and 2 we find the relation between the systems of

polynomials $\{\overline{P}_n(x)\}$ and $\{P_n(x)\}$, orthogonal to distribution functions $\overline{\rho}(x)$ and $\rho(x)$, in the following two cases:

a) when $\overline{\rho}(x)$ and $\rho(x)$ are related to one another as follows:

$$\overline{\rho}(x) = \int_0^x R(s)\, d\rho(s),$$

where $R(x)$ is an arbitrary rational function[1]; b) when the difference $\overline{\rho}(x) - \rho(x)$ is a stepped function with a finite number of steps. (Section 1 is a detailed presentation of the ideas first mentioned in our earlier paper[1]).

The relation between the systems of polynomials $\{\overline{P}_n(x)\}$ and $\{P_n(x)\}$ is given explicitly [cf. equations (18) and (26)].

In the theory of orthogonal polynomials a considerable part is played by functions of the second kind :

$$Q_n(x) = \int_{-\infty}^{\infty} \frac{P_n(s)\, d\rho(s)}{s - x}.$$

In Section 1 we derive the relation of second order functions $\{\overline{Q}_n(x)\}$ and $\{Q_n(x)\}$ which is obtained from the relation of the corresponding orthogonal polynomials $\{\overline{P}_n(x)\}$ and $\{P_n(x)\}$.

We note that the relation between polynomials $\{\overline{P}_n(x)\}$ and $\{P_n(x)\}$ in the first case, if the rational function $R(x)$ is a

polynomial, is known in the literature as the Christoffel formula because Christoffel was the first to derive it for $\rho(x) = x$ (ref. 2). We shall, therefore, call the relation between $\{\bar{P}_n(x)\}$ and $\{P_n(x)\}$ in the case when $R(x)$ is an arbitrary rational function the generalized Christoffel formula. Krein[3] developed a continuous analog of the Christoffel formula, and for one special case a continuous analog of the generalized Christoffel formula. In the general case a continuous analog of the generalized Christoffel formula was derived in ref. 4. A continuous analog of the relation between orthogonal polynomials $\{\bar{P}_n(x)\}$ and $\{P_n(x)\}$ is also known for the case when $\bar{\rho}(x) - \rho(x)$ is a step function[5].

The general method for obtaining continuous analogs of formulas from the theory of orthogonal polynomials was presented in detail in ref. 6.

1. General Christoffel formula

Let us recall some points of the theory of orthogonal polynomials[2].

1. Orthogonal polynomials $\{P_n(x)\}$ satisfy the recurrent relations

$$xP_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x),$$

Here

$$\alpha_n = a_n / a_{n+1},\ \beta_n = b_n / a_n - b_{n+1} / a_{n+1},\ \gamma_n = (a_{n-1} / a_n) d_n^2 / d_{n-1}^2,$$

a_n and b_n being the coefficients in the expansion

$P_n(x) = a_n x^n + b_n x^{n-1} + \ldots$, and

$$d_n^2 = \int_{-\infty}^{\infty} P_n^2(x)\, d\rho(x).$$

2. Functions of the second kind

$$Q_n(x) = \int_{-\infty}^{\infty} \frac{P_n(s)\, d\rho(s)}{s - x}$$

satisfy for $n \geqslant 1$ the same recurrent relations as polynomials $\{P_n(x)\}$, and

$$(a_n / a_{n+1} d_n^2)\,[Q_{n+1}(x) P_n(x) - Q_n(x) P_{n+1}(x)] = 1.$$

3. It is easy to derive from the recurrent relation the Darboux-Christoffel expression

$$\sum_{k=0}^{n} \frac{P_k(x) P_k(y)}{d_k^2} = \frac{a_n}{a_{n+1} d_n^2} \frac{P_{n+1}(x) P_n(y) - P_n(x) P_{n+1}(y)}{x - y}.$$

In this section we shall deal with the solution of the following problem: let $\{P_n(x)\}$ be a system of polynomials orthogonal in the interval $(-\infty, \infty)$ with respect to the distribution function $\rho(x)$. We are to find the connection between polynomials $\{P_n^{(k,l)}(x)\}$, orthogonal

in the same interval with respect to the distribution function $\rho_{k,l}(x)$, and polynomials $\{P_n(x)\}$, when function $\rho_{k,l}(x)$ is related to $\rho(x)$ by :

$$\rho_{k,l}(x)=\int_0^x\Big[\prod_{j=1}^{l}(s-\alpha_j)\Big]\Big[\prod_{j=1}^{k}(s-\beta_j)\Big]^{-1}d\rho(s). \qquad (1)$$

It is assumed here that integral (1) is meaningful. While solving this problem, we do not need to assume that functions $\rho(x)$ and $\rho_{k,l}(x)$ are monotonically non-diminishing and numbers α_j and β_j are real; all that needs to be assumed is that

$$d_n^2=\int_{-\infty}^{\infty}P_n^2(x)\,d\rho(x)\neq 0.$$

Lemma 1. In formula (1) let either $k = 1$ and $l = 0$ or $k = 0$ and $l = 1$. The polynomials $P_n^{(k,l)}(x)$ are then related to polynomials $P_n(x)$ by the following equations

$$P_n^{(0,1)}(x)=A_n\frac{P_{n+1}(x)P_n(\alpha_1)-P_n(x)P_{n+1}(\alpha_1)}{x-\alpha_1}, \qquad (2)$$

$$P_n^{(1,0)}(x)=B_n[P_n(x)Q_{n-1}(\beta_1)-P_{n-1}(x)Q_n(\beta_1)]. \qquad (3)$$

Here A_n and B_n are normalizing constants, and $Q_n(x)$ is a function of the second kind.

Proof. We examine the case when $k = 0$ and $l = 1$. We expand polynomial $P_n^{(0,\,1)}(x)$ in polynomials $P_n(x)$:

$$P_n^{(0,1)}(x) = \sum_{s=0}^{n} C_{n,s} P_s(x),$$

where

$$C_{n,s} = \frac{1}{d_s^2} \int_{-\infty}^{\infty} P_n^{(0,1)}(x) P_s(x)\, d\rho(x).$$

We transform the expression for $C_{n,s}$ as follows :

$$C_{n,s} = \frac{1}{d_s^2}\left\{\int_{-\infty}^{\infty} P_n^{(0,1)}(x)[P_s(x) - P_s(a_1)]\, d\rho(x) + \right.$$
$$\left. + P_s(a_1) \int_{-\infty}^{\infty} P_n^{(0,1)}(x)\, d\rho(x)\right\} = \frac{1}{d_s^2}\left[\int_{-\infty}^{\infty} P_n^{(0,1)}(x)(x - a_1) q_{s-1}(x)\, d\rho(x) + \right.$$
$$\left. + P_s(a_1) \int_{-\infty}^{\infty} P_n^{(0,1)}(x)\, d\rho(x)\right].$$

Here $q_{s-1}(x) = [P_s(x) - P_s(a_1)] / (x - a_1)$ is a polynomial of order s-1.

Since it is assumed that

$$\int_{-\infty}^{\infty} P_n^{(0,1)}(x) x^m\, d\rho_{0,1}(x) = \int_{-\infty}^{\infty} P_n^{(0,1)}(x) x^m (x - a_1)\, d\rho(x) = 0$$

when $m < n$, the first term in our formula is equal to zero, i.e.

$$C_{n,s} = \frac{P_s(a_1)}{d_s^2} \int_{-\infty}^{\infty} P_n^{(0,1)}(x)\, d\rho(x) = \bar{A}_n \frac{P_s(a_1)}{d_s^2}.$$

Finally, we find :

$$P_n^{(0,1)}(x) = \bar{A}_n \sum_{s=0}^{n} \frac{P_s(x) P_s(\alpha_1)}{d_s^2}.$$

Using the Darboux-Christoffel formula it is easily shown that the above expression is equivalent to (2).

Now suppose that $k = 1$ and $l = 0$. We then have

where

$$P_n^{(1,0)}(x) = \sum_{s=0}^{n} C_{n,s} P_s(x),$$

$$C_{n,s} = \frac{1}{d_s^2} \int_{-\infty}^{\infty} \frac{P_n^{(1,0)}(x)}{x - \beta_1} P_s(x)(x - \beta_1) d\rho(x).$$

Since

$$\int_{-\infty}^{\infty} P_n^{(1,0)}(x) x^m d\rho_{1,0}(x) = \int_{-\infty}^{\infty} \frac{P_n^{(1,0)}(x)}{x - \beta_1} x^m d\rho(x) = 0$$

when $m < n$, we have $C_{n,s} = 0$ when $s < n - 1$, i.e.

$$P_n^{(1,0)}(x) = C_{n,n} P_n(x) + C_{n,n-1} P_{n-1}(x). \tag{4}$$

Since

$$\int_{-\infty}^{\infty} \frac{P_n^{(1,0)}(x)}{x - \beta_1} d\rho(x) = 0,$$

we have from (4) : $C_{n,n} Q_n(\beta_1) + C_{n;n-1} Q_{n-1}(\beta_1) = 0$, i.e.

$C_{n,n} = B_n Q_n(\beta_1)$, $C_{n,n-1} = -B_n Q_n(\beta_1)$ where B_n is a constant. When these expressions are substituted in (4), we arrive at (3). The lemma

is thus proved.

With the aid of this lemma, it may be pointed out that generally it is worthwhile to try to obtain polynomials $P_n^{(k,l)}(x)$ in the form:

$$P_n^{(k,l)}(x) = \left[\prod_{j=1}^{l}(x-\alpha_j)\right]^{-1}\sum_{s=0}^{n+l} C_s P_s(x).$$

Here

$$C_s = \frac{1}{d_s^2}\int_{-\infty}^{\infty} P_n^{(k,l)}(x)\prod_{j=1}^{l}(x-\alpha_j)P_s(x)\,d\rho(x).$$

We transform the expression for C_s in the following manner:

$$C_s = \frac{1}{d_s^2}\int_{-\infty}^{\infty} P_n^{(k,l)}(x)P_s(x)\prod_{j=1}^{k}(x-\beta_j)\,d\rho_{k,l}(x).$$

In view of the fact that the polynomials $P_n^{(k,l)}(x)$ are orthogonal, the coefficients $C_s = 0$ when $s < n - k$, i.e. $P_n^{(k,l)}(x)$, when $n \geq k$, may be validly expanded into

$$P_n^{(k,l)}(x) = \left[\prod_{j=1}^{l}(x-\alpha_j)\right]^{-1}\sum_{s=n-k}^{n+l} C_s P_s(x). \qquad (5)$$

(the case of $n < k$ will be considered separately).

It is evident that

$$\sum_{s=n-k}^{n+l} C_s P_s(\alpha_j) = 0 \qquad (j = 1, 2, \ldots, l). \qquad (6)$$

The still missing equations required to determine the expansion coefficients k can be obtained from the orthogonality conditions

$$\int_{-\infty}^{\infty} P_n^{(k,l)}(x) \prod_{\substack{j=1 \\ j\neq i}}^{k} (x-\beta_j)\, d\rho_{k,l}(x) = 0 \quad (i = 1, 2, \ldots, k).$$

Substituting into these equalities the expressions (5) and (1), we have a set of equations :

$$\sum_{s=n-k}^{n+l} C_s Q_s(\beta_i) = 0 \quad (i = 1, 2, \ldots, k). \tag{7}$$

Eliminating C_s from (5) - (7) and regarding C_{n+l} as known, we obtain

$$P_n^{(k,l)}(x) = A_n^{(k,l)} \left[\prod_{j=1}^{l} (x-\alpha_j) \right]^{-1} \begin{vmatrix} P_{n-k}(\alpha_1) & \ldots & P_{n+l}(\alpha_1) \\ \ldots & \ldots & \ldots \\ P_{n-k}(\alpha_l) & \ldots & P_{n+l}(\alpha_l) \\ Q_{n-k}(\beta_1) & \ldots & Q_{n+l}(\beta_1) \\ \ldots & \ldots & \ldots \\ Q_{n-k}(\beta_k) & \ldots & Q_{n+l}(\beta_k) \\ P_{n-k}(x) & \ldots & P_{n+l}(x) \end{vmatrix}, \tag{8}$$

where $A_n^{(k,l)} = C_{n+l}/\Delta$ is a normalizing coefficient with

$$\Delta = \begin{vmatrix} P_{n-k}(\alpha_1) & \ldots & P_{n+l-1}(\alpha_1) \\ \ldots & \ldots & \ldots \\ P_{n-k}(\alpha_l) & \ldots & P_{n+l-1}(\alpha_l) \\ Q_{n-k}(\beta_1) & \ldots & Q_{n+l-1}(\beta_1) \\ \ldots & \ldots & \ldots \\ Q_{n-k}(\beta_k) & \ldots & Q_{n+l-1}(\beta_k) \end{vmatrix}. \tag{9}$$

We still have to find the relation between polynomials $P_n^{(k,l)}(x)$ and $P_n(x)$ in the case of $n < k$.

When $n < k$, we shall try to find $P_n^{(k,l)}$ in a form, similar to (8), namely

$$P_n^{(k,l)}(x) = A_n^{(k,l)}\left[\prod_{j=1}^{l}(x-\alpha_j)\right]^{-1}\Delta(a,b,c,P(\alpha),Q(\beta),P(x)), \tag{10}$$

where

$$\Delta(a,b,c,P(\alpha),Q(\beta),P(x)) = \begin{vmatrix} a_{1,1}\ldots a_{1,k-n}\,P_0(\alpha_1)\ldots P_{n+l}(\alpha_1) \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{l,1}\ldots a_{l,k-n}\,P_0(\alpha_l)\ldots P_{n+l}(\alpha_l) \\ b_{1,1}\ldots b_{1,k-n}\,Q_0(\beta_1)\ldots Q_{n+l}(\beta_1) \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ b_{k,1}\ldots b_{k,k-n}\,Q_0(\beta_k)\ldots Q_{n+l}(\beta_k) \\ c_1 \quad\ldots\quad c_{k-n}\,P_0(x)\ldots P_{n+l}(x) \end{vmatrix}.$$

Here a, b, c, $P(\alpha)$, $Q(\beta)$, $P(x)$ are matrices of corresponding size and $A_n^{(k,l)}$, $a_{i,k}$, $b_{i,k}$, and c_k are constants. It is obvious that determinant (10) must be fully (without remainder) divisible by $\prod_{j=1}^{l}(x-\alpha_j)$. This can be achieved if it is assumed that $a_{j,k} = c_k$ for any value of i. Allowing for the properties of the determinant, (10) can be rewritten as

$$P_n^{(k,l)}(x) = A_n^{(k,l)}\Delta(0,\bar{b},0,P(\alpha),Q(\beta),P(x))\left[\prod_{j=1}^{l}(x-\alpha_j)\right]^{-1}, \tag{11}$$

where $\bar{b}_{ik}$ are some new constants.

The orthogonality relations

$$\int_{-\infty}^{\infty} P_n^{(k,l)}(x)\, x^m\, d\rho_{k,l}(x) = 0, \qquad m < n,$$

are fulfilled for $P_n^{(k,l)}(x)$ if

$$\int_{-\infty}^{\infty} P_n^{(k,l)}(x) \prod_{j=1}^{m} (x-\beta_j)\, d\rho_{k,l}(x) = 0, \qquad m < n. \tag{12}$$

It follows from the condition (12) that when $m < n$

$$\Delta[0, \bar{b}, 0, P(\alpha), Q(\beta), R] = 0, \tag{13}$$

where

$$R = \left\{ \int_{-\infty}^{\infty} [P_0(x)\, d\rho(x)] \left[\prod_{j=m+1}^{k} (x-\beta_j) \right]^{-1}, \ldots \right.$$

$$\left. \ldots, \int_{-\infty}^{\infty} [P_{n+l}(x)\, d\rho(x)] \left[\prod_{j=m+1}^{k} (x-\beta_j) \right]^{-1} \right\}.$$

We shall now rewrite the left-hand side of (13) by expanding $\left[\prod_{j=m+1}^{k} (x-\beta_j) \right]^{-1}$ into common fractions:

$$\left[\prod_{j=m+1}^{k} (x-\beta_j) \right]^{-1} = \sum_{j=m+1}^{k} \frac{a_j}{x-\beta_j}. \tag{14}$$

From the above

$$\int_{-\infty}^{\infty} [P_s(x)\, d\rho(x)] \left[\prod_{j=m+1}^{k} (x-\beta_j) \right]^{-1} = \sum_{j=m+1}^{k} a_j Q_s(\beta_j). \tag{15}$$

Coefficients a_j satisfy the following conditions :

$$\sum_{j=m+1}^{k} a_j \beta_j^s = \begin{cases} 0, & 0 \leqslant s \leqslant k-m-2, \\ 1, & s = k-m-1. \end{cases} \tag{16}$$

This can be proved by solving the set of equations (16) with respect to coefficients a_j and comparing the results with the expressions for coefficients a_j obtained from expansion (14).

Using (15) and allowing for the well known properties of determinants the left-hand side of (13) can be written in the form :

$$\Delta(0, \bar{b}, \bar{c}, P(\alpha), Q(\beta), 0) = 0, \tag{17}$$

where

$$\bar{c} = \left(- \sum_{j=m+1}^{k} a_j \bar{b}_{j,1}, \ldots, - \sum_{j=m+1}^{k} a_j \bar{b}_{j,k-n} \right).$$

The characteristic features (16) of coefficients a_j make it clear that conditions (17) will be satisfied if we put $\bar{b}_{j,s} = \beta_j^{s-1}$

We have thus proved

Theorem 1. For any values of k and l, if there are no identical values among constants α_j (or β_j), with $n < k$ the following formula is valid for $P_n^{(k,l)}(x)$:

$$P_n^{(k,l)} = A_n^{(k,l)} \Big[\prod_{j=1}^{l} (x - \alpha_j) \Big]^{-1} \Delta(0, \bar{\beta}, 0, P(\alpha), Q(\beta), P(x)), \tag{18}$$

where $\bar{\beta} = \|b_{ij}\|$, $b_{ij} = \beta_i^{j-1}$, $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant k-n$, and when $n \geqslant k$, the formula which is valid is (8). ($A_n^{(k,l)}(x)$ is a normalizing constant).

From (8), (18), and (14) (for $m = 0$) it is quite easy to derive also the formulas for functions of the second kind:

$$Q_n^{(k,l)}(x) = \int_{-\infty}^{\infty} \frac{P_n^{(k,l)}(s)\, d\rho_{k,l}(s)}{s - x}.$$

For $n < k$,

$$Q_n^{(k,l)}(x) = A_n^{(k,l)} \Big[\prod_{j=1}^{k} (x - \beta_j) \Big]^{-1} \Delta(0, \bar{\beta}, \bar{x}, P(\alpha), Q(\beta), Q(x)),$$

where $\bar{x} = (x^0, \ldots\ldots, x^{k-n-1})$; when $n \geqslant k$, the expression for functions of the second kind is obtained from (8) by substituting $P_i(x)$, $n - k \leqslant i \leqslant n + 1$, and the product $\prod_{j=1}^{l} (x-\alpha_j)$ by $\prod_{j=1}^{k} (x-\beta_j)$.

In the simplest case, when $k = 0$, $\rho(x) = x$, expression (8) was developed by Christoffel. For this reason we shall from now on call formulas (18) and (8) generalized Christoffel formulas.

Note. Using a recurrent relation for polynomials $P_n(x)$ and the second kind functions $Q_n(x)$, the generalized Christoffel formulas can be written in another form by replacing in (8) the

$P_{n-k+2s}(x)$, $P_{n-k+2s}(\alpha)$, $Q_{n-k+2s}(\beta)$ by $x^s P_{n-k+s}(x)$, $\alpha^s P_{n-k+s}(\alpha)$, $\beta^s Q_{n-k+s}(\beta)$, and also replacing $P_{n-k+2s+1}(x)$, $P_{n-k+2s+1}(\alpha)$, $Q_{n-k+2s+1}(\beta)$ by $x^s P_{n-k+s+1}(x)$, $\alpha^s P_{n-k+s+1}(\alpha)$ and $\beta^s Q_{n-k+s+1}(\beta)$. Expression (18) can also be simplified by replacing in it $P_s(x)$, $P_s(\alpha)$, $Q_s(\beta)$ by x^s, α^s, and $\int_{\infty}^{\infty} [x^s/(x-\beta)]d\rho(x)$ respectively.

For the case when some of the values of α_j (or β_j) are identical, the generalized Christoffel formulas can be easily obtained from (18) and (8). For instance, let $\alpha_1 = \alpha_2$ in (8) and let all the other values of α_j and β_j be different. In this case, taking the normalizing constant as $A_n/(\alpha_2-\alpha_1)$, we obtain

$$P_n^{(k,l)}(x) = \lim_{\alpha_1\to\alpha_2} A_n\left[(\alpha_2-\alpha_1)\prod_{j=1}^{l}(x-\alpha_j)\right]^{-1}\Delta(P(\alpha),Q(\beta),P(x)) =$$

$$= \lim_{\alpha_1\to\alpha_2} A_n\left[\prod_{j=1}^{l}(x-\alpha_j)\right]^{-1}\Delta(P_1(\alpha),Q(\beta),P(x)) =$$

$$= A_n\left[\prod_{j=1}^{l}(x-\alpha_j)\right]^{-1}\Delta(P'(\alpha),Q(\beta),P(x)),$$

where $\Delta(P(\alpha), Q(\beta), P(x))$ denotes the determinant in (8): the change from $P(\alpha)$ to $P_1(\alpha)$ means that in the second row instead of $P_{n-k+j}(\alpha_2)$ is taken $[P_{n-k+j}(\alpha_2) - P_{n-k+j}(\alpha_1)]/(\alpha_2 - \alpha_1)$; similarly, $P'(\alpha)$ signifies that $P_{n-k+j}(\alpha_1)$ is taken instead of $P_{n-k+j}(\alpha_2)$.

In a similar manner we can arrive at the following result : if any values of α_i and α_j (or β_i and β_j) are identical, it is necessary to replace in the j-th row of the determinant, defining the polynomial $P_n^{(k,l)}(x)$, the functions themselves by their derivatives. If several values of α_i (or β_i) are identical, the determinant must have, besides, the row with the first derivatives, also rows with the second, third, etc. derivatives (right up to derivatives whose order is equal to the number of identical values of α_i (or β_i)).

2. Relation between systems of polynomials orthogonal with respect to distribution functions which differ by a piecewise constant function

Let distribution functions $\bar{\rho}(\lambda)$ and $\rho(\lambda)$ be related by :

$$\bar{\rho}(\lambda) = \rho(\lambda) + \sum_{j=1}^{p} m_j e(\lambda - \lambda_j), \tag{19}$$

where

$$e(\lambda)=\begin{cases}0, & \lambda<0,\\ 1, & \lambda\geqslant 0.\end{cases}$$

In order to find the relation between polynomial $\overline{P}_n(\lambda)$ and polynomials $P_n(\lambda)$, we expand $\overline{P}_n(\lambda)$ in polynomials $P_n(\lambda)$:

$$\overline{P}_n(\lambda)=\sum_{s=0}^{n} C_s P_s(\lambda). \tag{20}$$

Using the orthogonality conditions of polynomials $\overline{P}_s(\lambda)$

$$\int_{-\infty}^{\infty} \overline{P}_n(\lambda) P_s(\lambda)\, d\bar{\rho}(\lambda)=0, \qquad s<n,$$

we obtain

$$C_s d_s^2+\sum_{j=1}^{p} m_j \overline{P}_n(\lambda_j) P_s(\lambda_j)=0, \qquad s<n. \tag{21}$$

Substituting the coefficients C_s obtained from (21) into expansion (20), we obtain :

$$\overline{P}_n(\lambda)=-\sum_{j=1}^{p} m_j \overline{P}_n(\lambda_j) a_j(\lambda)+C_n P_n(\lambda), \tag{22}$$

where

$$a_j(\lambda)=\sum_{s=0}^{n-1}\frac{P_s(\lambda)P_s(\lambda_j)}{d_s^2}=\frac{a_{n-1}}{a_n d_{n-1}^2}\,\frac{P_n(\lambda)P_{n-1}(\lambda_j)-P_{n-1}(\lambda)P_n(\lambda_j)}{\lambda-\lambda_j}$$

To obtain the final expression for $\overline{P}_n(\lambda)$ it is necessary to find quantities $\overline{P}_n(\lambda_j)$. To do this we put in (22) $\lambda=\lambda_i$, $i=1,2,\ldots,p$.

We then have the following system of equations for $\bar{P}_n(\lambda_j)$:

$$\bar{P}_n(\lambda_i) + \sum_{j=i}^{n} m_j a_{ij} \bar{P}_n(\lambda_j) = C_n P_n(\lambda_i), \quad i = 1, 2, \ldots, p, \tag{23}$$

where

$$a_{ij} = a_j(\lambda_i).$$

Eliminating $\bar{P}_n(\lambda_j)$ from (22), (23) we have finally :

$$\bar{P}_n(\lambda) = \bar{c}_n \det \|\Delta_{ij}\|, \quad 1 \leqslant i, j \leqslant p+1,$$

where for $1 \leqslant i, j \leqslant p$, we have

$$\Delta_{ij} = a_{ij} m_j, \ i \neq j; \ \Delta_{ii} = 1 + a_{ii} m_i; \ \Delta_{i, p+1} = P_n(\lambda_i); \ \Delta_{p+1, j} = a_j(\lambda) m_j.$$

Further,

$$\Delta_{p+1, p+1} = P_n(\lambda), \quad \bar{c}_n = c_n / \Delta -$$

is the normalizing coefficient, and

$$\Delta = \det \| \delta_{ij} + a_{ij} m_j \|.$$

Received February 9, 1968

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*Translator's note : Spelling Uncertain.